

Average Entropy of a Subsystem

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It was recently conjectured by D. Page that if a quantum system of Hilbert space dimension nm is in a random pure state then the average entropy of a subsystem of dimension m where $m \leq n$ is $S_{mn} = \sum_{k=n+1}^{mn} (1/k) - (m-1)/2n$. In this letter this conjecture is proved.

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In a recent letter Page [1] considered a system AB with Hilbert space dimension mn . The system consisted of two subsystems A and B of dimensions m and n respectively. Page calculated the average

$$S_{mn} = \langle S_A \rangle$$

of the entropy S_A over all pure states $\rho = |\Psi\rangle\langle\Psi|$ of the total system where $S_A = -\text{Tr } \rho_A \ln \rho_A$ and ρ_A , the density matrix of subsystem A , is obtained by taking the partial trace of the full density matrix ρ over the other subsystem, that is, $\rho_A = \text{Tr}_B \rho$.

The average was defined with respect to the unitary invariant Haar measure on the space of unitary vectors $|\Psi\rangle$ in the mn dimensional Hilbert space of the total system. The quantity $(\ln m - S_{mn})$ was used to define the average information of the subsystem A . It is a measure of the information regarding the fact that the entire system is a pure state that is contained in the subsystem m . Using earlier work [2,3] in this area, Page was led to consider the probability distribution of the eigenvalues of ρ_A for the random pure states ρ of the entire system. He used

$$P(p_1, \dots, p_m) dp_1 \dots dp_m \\ = N \delta(1 - \sum_{i=1}^m p_i) \prod_{1 \leq i < j \leq m} (p_i - p_j)^2 \prod_{k=1}^m p_k^{n-m} dp_k$$

where p_i was an eigenvalue of ρ_A and the normalisation constant for this probability distribution was given only implicitly by the requirement that the total probability integrated to unity. Page then showed that the average

$$S_{mn} = - \int \left(\sum_{i=1}^m p_i \ln p_i \right) P(p_1, \dots, p_m) dp_1 \dots dp_m \\ = \psi(mn+1) - \frac{\int (\sum_{i=1}^m q_i \ln q_i) Q dq_1 \dots dq_m}{mn \int Q dq_1 \dots dq_m}$$

where $q_i = r p_i$ for $i = 1, \dots, m$, r is positive [1], and

$$\psi(mn+1) = -C + \sum_{k=1}^{mn} \frac{1}{k},$$

C being Euler's constant, and

$$Q(q_1, \dots, q_m) dq_1 \dots dq_m = \prod_{1 \leq i < j \leq m} (q_i - q_j)^2 \prod_{i=1}^m e^{-q_i} q_i^{n-m} dq_i.$$

On the basis of evaluating S_{mn} for $m = 2, 3, 4, 5$ using MATHEMATICA 2.0, Page conjectured that the exact result for S_{mn} was

$$S_{mn} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{(m-1)}{2n},$$

but was not able to prove that this was the case. In this letter, we will prove this conjecture.

We first observe that the van der Monde determinant defined by

$$\Delta(q_1, \dots, q_m) \equiv \prod_{i \leq j \leq m} (q_i - q_j)$$

may be written

$$\Delta(q_1, \dots, q_m) = \begin{vmatrix} 1 & \dots & 1 \\ q_1 & \dots & q_m \\ \vdots & \ddots & \vdots \\ q_1^{m-1} & \dots & q_m^{m-1} \end{vmatrix}.$$

We next observe that $\Delta(q_1, \dots, q_m)$ can be written as

$$\Delta(q_1, \dots, q_m) = \begin{vmatrix} p_0(q_1) & \dots & p_0(q_m) \\ p_1(q_1) & \dots & p_1(q_m) \\ \vdots & \ddots & \vdots \\ p_{m-1}(q_1) & \dots & p_{m-1}(q_m) \end{vmatrix} \quad (1)$$

for any set of polynomials $p_k(q)$, $k = 0, \dots, m-1$, which have the property, $p_0(q) = 1$, and

$$p_k(q) = q^k + C_{k-1} q^{k-1} + \dots + C_0, \quad k = 1, \dots, m-1.$$

This immediately follows from the fact that the value of a determinant does not change if the multiple of any one row is added to a different row.

We now choose polynomials $p_k^\alpha(q)$ judiciously. We introduce orthogonal polynomials $p_k^\alpha(q)$ with the properties:

$$1. \quad p_k^\alpha(q) = q^k + C_{k-1}^\alpha q^{k-1} + \dots + C_0^\alpha, \quad p_0^\alpha(q) = 1.$$

$$2. \int_0^\infty dq e^{-q} q^\alpha p_{k_1}^\alpha(q) p_{k_2}^\alpha(q) = h_{k_1}^\alpha \delta_{k_1 k_2}, \quad \alpha = n - m.$$

Polynomials with these properties are well known. They are the generalised Laguerre polynomials defined by [4]

$$p_k^\alpha(q) = \frac{e^q}{q^\alpha} (-1)^k \frac{d^k}{dq^k} (e^{-q} q^{k+\alpha}).$$

We also note, for later use, that [4]

$$p_k^\alpha(q) = \sum_{r=0}^k \binom{k}{r} (-1)^r \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-r+1)} q^{k-r} \quad (2)$$

$$\int_0^\infty dq e^{-q} q^\alpha p_{k_1}^\alpha(q) p_{k_2}^\alpha(q) = \Gamma(k_1+1) \Gamma(k_1+\alpha+1) \delta_{k_1 k_2} \quad (3)$$

$$\int_0^\infty dq q^{a-1} e^{-q} p_k^b(q) = (1-a+b)_k \Gamma(a) (-1)^k \quad (4)$$

recalling that, $(1-a+b)_k = (1-a+b)(1-a+b+1)\dots(1-a+b+k-1)$. Writing $\Delta(q_1, \dots, q_m)$ in terms of $p_k^\alpha(q)$ as in Eq. 1 and using the orthogonal property of these polynomials it immediately follows that:

$$S_{mn} = \psi(mn+1) - \frac{1}{mn} \sum_{k=0}^{m-1} \int_0^\infty \frac{e^{-q} (q \ln q) q^{n-m} (p_k^{m-n}(q))^2 dq}{\Gamma(k+1) \Gamma(k+1+n-m)}.$$

We thus need to evaluate the integral

$$I_{nm}^k = \int_0^\infty (q \ln q) q^{n-m} (p_k^{m-n}(q))^2 e^{-q} dq.$$

We first introduce

$$J^k(\alpha) = \int_0^\infty q^{\alpha+1} (p_k^\alpha(q))^2 e^{-q} dq.$$

This integral is easily evaluated. We have

$$J^k(\alpha) = \Gamma(k+1) \Gamma(k+\alpha+2) + k^2 \Gamma(k) \Gamma(k+\alpha+1) \quad (5)$$

and we now note that

$$I_{nm}^k = \left[\frac{dJ^k(\alpha)}{d\alpha} - 2 \int_0^\infty dq q^{\alpha+1} e^{-q} p_k^\alpha \frac{dp_k^\alpha}{d\alpha} \right]_{\alpha=n-m}.$$

Evaluating these two terms using Eqs. (2), (3), (4), and (5) we find

$$\begin{aligned} S_{mn} &= \psi(mn+1) \\ &- \frac{1}{mn} \sum_{k=0}^{m-1} [1 + (1+2k+n-m)\psi(k+n-m+1)] \\ &+ \frac{2}{mn} \sum_{k=0}^{m-1} \sum_{r=0}^k \binom{k}{r} (-1)^{k+r} \frac{\Gamma(k+n-m+1)}{\Gamma(k+n-m-r+1)} \\ &\times [\psi(k+n-m+1) - \psi(k+n-m-r+1)] \\ &\times \frac{(r-k-1)_k \Gamma(k+n-m-r+2)}{\Gamma(k+1) \Gamma(k+n-m+1)} \end{aligned} \quad (6)$$

where we use the fact that $\psi(z) = \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz}$. We now observe that

$$\begin{aligned} &\psi(mn+1) \\ &- \frac{1}{mn} \sum_{k=0}^{m-1} [1 + (1+2k+n-m)\psi(1+k+n-m)] \\ &= \sum_{k=n+1}^{mn} \frac{1}{k} + \frac{(m-1)}{2n}. \end{aligned} \quad (7)$$

This follows by examining the coefficient of $\frac{1}{r}$ in

$$\sum_{k=0}^{m-1} (1+2k+n-m)\psi(1+k+n-m)$$

and writing

$$\psi(1+k+n-m) = -C + \sum_{r=1}^{k+n-m} \frac{1}{r}.$$

The third expression in Eq. (6) above is

$$\begin{aligned} &\frac{2}{mn} \sum_{k=0}^{m-1} \sum_{r=0}^k \binom{k}{r} (-1)^{k+r} \frac{\Gamma(k+n-m+1)}{\Gamma(k+n-m-r+1)} \\ &\times [\psi(k+n-m+1) - \psi(k+n-m-r+1)] \\ &\times \frac{(r-k-1)_k \Gamma(k+n-m-r+2)}{\Gamma(k+1) \Gamma(k+n-m+1)} \\ &= \frac{2}{mn} \sum_{k=0}^{m-1} \binom{k}{1} (-1)^{2k+1} \\ &= -2 \frac{(m-1)}{2n}. \end{aligned} \quad (8)$$

Since $(r-k-1)_k = 0$, for all $r \neq 0$ and $r \neq 1$, and also $\psi(k+n-m+1) - \psi(k+n-m-r+1) = 0$ when $r = 0$, we obtain

$$S_{mn} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{(m-1)}{2n} \quad (9)$$

as conjectured by Page.

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